Midterm Exam - Advanced Linear Algebra
M. Math II

22 September, 2022

(i) Duration of the exam is 3 hours.
(ii) The maximum number of points you can score in the exam is 100.
(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. A matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ is said to be *unistochastic* if there is a unitary matrix $(u_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{C})$ such that $a_{ij} = |u_{ij}|^2$.
   (a) (10 points) Show that the doubly stochastic matrix $\mathbf{J}_n \in M_n(\mathbb{R})$ which has all its entries identically equal to $\frac{1}{n}$, is a unistochastic matrix.
   (b) (10 points) With justification, give an example of a doubly stochastic matrix in $M_3(\mathbb{R})$ which is **not** unistochastic.

Total for Question 1: 20

2. (a) (10 points) Show that the set of extreme points of the convex set $\mathrm{ext}\big(\{A \in M_n(\mathbb{C}) : \|A\| \leq 1\}\big)$, is the set of $n \times n$ unitary matrices.
   (b) (10 points) Let $A \in M_n(\mathbb{C})$ with $\|A\| \leq 1$. Show that there are unitaries $U_1, U_2 \in U_n(\mathbb{C})$ such that
$$A = \frac{U_1 + U_2}{2}.$$

Total for Question 2: 20

3. Let $\Phi : M_n(\mathbb{C}) \to D_n(\mathbb{C})$ be the unique trace-preserving conditional expectation onto the $*$-algebra of diagonal matrices.

   (a) (10 points) For a symmetric and positive-semidefinite matrix $H \in M_n(\mathbb{C})$, show that $\lambda^{\downarrow}(\Phi(H)) \prec \lambda^{\downarrow}(H)$, where $\lambda^{\downarrow}(\cdot)$ denotes the vector of eigenvalues (counted with multiplicity) arranged in decreasing order.

   (b) (15 points) Show that $\Phi(A)^*\Phi(A) \leq \Phi(A^*A)$ for all $A \in M_n(\mathbb{C})$. Using this, prove that for $1 \leq k \leq n$, we have

$$\sum_{j=1}^{k} s_j(\Phi(A))^2 \leq \sum_{j=1}^{k} s_j(A)^2,$$

   where $s_j(\cdot)$ denotes the jth singular value.

Total for Question 3: 25

4. (a) (10 points) If $\{\lambda_j^{\downarrow}(A)\}$ denotes the eigenvalues of an Hermitian matrix $A \in M_n(\mathbb{C})$ arranged in decreasing order, then for $1 \leq k \leq n$, show that we have

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max \sum_{j=1}^{k} \langle x_j, Ax_j \rangle,$$

   where the maximum is taken over all orthonormal k-tuples of vectors $(\vec{x}_1, \ldots, \vec{x}_k)$ in $\mathbb{C}^n$.

   (b) (10 points) Show that for $A, B \in M_n(\mathbb{C})$ and $1 \leq k \leq n$, we have

$$\sum_{j=1}^{k} s_j(A+B) \leq \sum_{j=1}^{k} s_j(A) + \sum_{j=1}^{k} s_j(B),$$

   where $s_j(\cdot)$ denotes the jth singular value.

Total for Question 4: 20

5. (15 points) Let $n \geq 2$. Let $B_n(\mathbb{R})$ denote the set of upper-triangular matrices in $M_n(\mathbb{R})$. Show that it is not possible to find continuous functions $Q : M_n(\mathbb{R}) \to O_n(\mathbb{R}), R : M_n(\mathbb{R}) \to B_n(\mathbb{R})$ such that $Q(A)R(A) = A$ for all $A \in M_n(\mathbb{R})$. (In other words, the QR-factorization cannot be done in a continuous manner over all of $M_n(\mathbb{R})$.)

Total for Question 5: 15